

GENERALIZED DERIVATIONS OF 3-LIE ALGEBRAS

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ABSTRACT. Generalized derivations, quasiderivations and quasicentroid of 3-algebras are introduced, and basic relations between them are studied. Structures of quasiderivations and quasicentroid of 3-Lie algebras, which contains a maximal diagonalized tours, are systematically investigated. The main results are: for all 3-Lie algebra A , 1) the generalized derivation algebra $GDer(A)$ is the sum of quasiderivation algebra $QDer(A)$ and quasicentroid $Q\Gamma(A)$; 2) quasiderivations of A can be embedded as derivations in a larger algebra; 3) quasiderivation algebra $QDer(A)$ normalizer quasicentroid, that is, $[QDer(A), Q\Gamma(A)] \subseteq Q\Gamma(A)$; 4) if A contains a maximal diagonalized tours T , then $QDer(A)$ and $Q\Gamma(A)$ are diagonalized T -modules, that is, as T -modules, (T, T) semi-simply acts on $QDer(A)$ and $Q\Gamma(A)$, respectively.

1. INTRODUCTION

3-Lie algebras [1] have close relationships with many important fields in mathematics and mathematical physics (cf. [2, 3, 4, 5, 6, 7, 8]). The multiple $M2$ -brane model of Bagger-Lambert [4] and Gustavsson [7] is defined on Lie 3-algebra, which serve as the gauge symmetry algebras for the $M2$ -brane world-volume theory. For the consistency of these symmetries, it need to impose the fundamental identity on the Lie 3-algebras. P. Ho, Y. Imamura, Y. Matsuo in paper [9] studied two derivations of the multiple $D2$ action from the multiple $M2$ -brane model proposed by Bagger-Lambert and Gustavsson. One defined 3-Lie algebra which contains generators of a given Lie algebra. Such an extension contains generators with negative norms. By suitably choosing such extension, one might restrict the field associated with it to constant or zero while keeping almost all of the symmetry of BLG theory. The second derivation of multiple $D2$ -brane, the extra generator has a simple physical origin, the winding of $M5$ -brane around S^1 which defines the reduction from M -theory to the type IIA theory. One may provide a similar geometrical origin to other 3-Lie algebras.

The concept of derivations appear in different mathematical fields with many different forms. In algebra systems, derivations are linear mappings satisfying the Leibniz relation. So several kinds of derivations, for example, triple derivations, generalized derivations, quasiderivations of algebras are studied [10, 11, 12, 13]. In n -Lie algebras derivations are very useful, derivations are also studied [14, 15, 16, 17]. It is proved that the Lie algebra of the autoisomorphism group $Aut(L)$ of an n -Lie algebra L is its derivation algebra $Der(L)$. And if M is an L -module,

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then M is also a module of its inner derivation algebra $\text{ad}(L)$. By means of this relation, finite dimensional irreducible modules of simple n -Lie algebras over the complex field is classified [18].

In this paper, we study the generalized derivations of 3-Lie algebras. Although definitions of generalized derivations, quasiderivations and quasicentroid on 3-Lie algebras are similar to the definitions on Lie algebras [19], but structures of them are very different (see the discussion in section 4 and section 5). In the following, \mathbb{F} denotes a field with $\text{char } \mathbb{F} \neq 2$ and 3 , $\text{Hom}(W, V)$ is a vector space spanned by all the linear maps from vector space W onto vector space V .

2. FUNDAMENTAL NOTIONS

A 3-Lie algebra is a vector space A over \mathbb{F} endowed with a 3-ary multi-linear skew-symmetric multiplication $[\cdot, \cdot, \cdot]$ satisfying following identity, for all $x_1, x_2, x_3, y_2, y_3 \in A$,

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^3 [x_1, [x_i, y_2, y_3], x_3]. \quad (2.1)$$

A *derivation* of a 3-Lie algebra A is an element $f \in \text{Hom}(A, A)$ such that for all $x, y, z \in A$,

$$[f(x), y, z] + [x, f(y), z] + [x, y, f(z)] = f([x, y, z]).$$

The set of all derivations of A is denoted by $\text{Der}(A)$, which is a subalgebra of the general linear Lie algebra $\text{gl}(A)$.

A left multiplication of A determined by $x_1, x_2 \in A$ is a linear map $\text{ad}(x_1, x_2) : A \rightarrow A$, defined by

$$\text{ad}(x_1, x_2)(x) = [x_1, x_2, x],$$

where $x \in A$. A left multiplication is a derivation. The linear combinations of left multiplications are called inner derivations; the set of all inner derivations is denoted by $\text{ad}(A)$, which is an ideal of $\text{Der}(A)$.

Let B_1, B_2, B_3 be subspaces of A . The notation $[B_1, B_2, B_3]$ denotes the subspace of A spanned by vectors $[x_1, x_2, x_3]$, where $x_i \in B_i$, $1 \leq i \leq 3$. A subspace B is called a *subalgebra* (an *ideal*) if B satisfies $[B, B, B] \subseteq B$ ($[B, A, A] \subseteq B$). In particular, the subalgebra generated by vectors $[x_1, x_2, x_3]$ for all $x_1, x_2, x_3 \in A$ is called the *derived algebra* of A , which is denoted by A^1 . If $A^1 = 0$, then A is called an *abelian algebra*.

The *center* of A is $Z(A) = \{x \in A \mid [x, A, A] = 0\}$. It is clear that $Z(A)$ is an abelian ideal of A .

A *centre derivation* of A (see [20]) is an element $f \in \text{Hom}(A, A)$ satisfying $f(A) \subseteq Z(A)$ and $f(A^1) = 0$. The set of all centre derivations is denoted by $\text{ZDer}(A)$, which is an ideal of $\text{Der}(A)$.

Definition 2.1 Let A be a 3-Lie algebra, f_1 be a linear map of A . If there exist linear maps f_2, f_3, f' of A satisfy for all $x, y, z \in A$,

$$[f_1(x), y, z] + [x, f_2(y), z] + [x, y, f_3(z)] = f'([x, y, z]), \quad (2.2)$$

then f_1 is called a generalized derivation of A . The set of all quaternions (f_1, f_2, f_3, f') is denoted by $\Delta(A)$, and the set of all generalized derivations is denoted by $GDer(A)$. In the case of $f_1 = f_2 = f_3 = f$, the linear map f is called a quasiderivation of A , and the set of all quasiderivations is denoted by $QDer(A)$.

By the above definition, for any 3-Lie algebra A , we have

$$\text{ad}(A) \subseteq Der(A) \subseteq QDer(A) \subseteq GDer(A).$$

And by the skew-symmetry of the multiplication, if $(f_1, f_2, f_3, f') \in \Delta(A)$, then

$$[f_2(x), y, z] + [x, f_1(y), z] + [x, y, f_3(z)] = f'([x, y, z]). \quad (2.3)$$

It follows that $f_1, f_2, f_3 \in GDer(A)$ and $(f_{i_1}, f_{i_2}, f_{i_3}, f') \in \Delta(A)$, where (i_1, i_2, i_3) is arbitrary 3-ary permutation.

We know that the *centroid* [20] of a 3-Lie algebra A is an associative algebra $\Gamma(A)$ composed of linear maps f satisfying for all $x, y, z \in A$,

$$[f(x), y, z] = [x, f(y), z] = [x, y, f(z)] = f([x, y, z]). \quad (2.4)$$

We give the generalization of $\Gamma(A)$ as follows.

Definition 2.2 Let A be a 3-Lie algebra. The quasicentroid of A , is denoted by $Q\Gamma(A)$, is a vector space spanned by all elements $f \in \text{Hom}(A, A)$ which satisfies for all $x, y, z \in A$,

$$[f(x), y, z] = [x, f(y), z] = [x, y, f(z)]. \quad (2.5)$$

3. GENERALIZED DERIVATIONS

In this section we study the relations between $GDer(A)$, $QDer(A)$, $Der(A)$, $Q\Gamma(A)$ and $\Gamma(A)$ of a 3-Lie algebra A . And we see that quasiderivations of A can be embedded as derivations in a larger algebra.

Theorem 3.1. Let A be a 3-Lie algebra. Then $GDer(A)$, $QDer(A)$, $Q\Gamma(A)$ and $\Gamma(A)$ are subalgebras of the general linear Lie algebra $gl(A)$. In addition, if $Z(A) = 0$, then $Q\Gamma(A)$ and $\Gamma(A)$ are abelian 3-Lie algebras.

Proof For arbitrary $f, g \in GDer(A)$ and $(f, f', f'', f'''), (g, g', g'', g''') \in \Delta(A)$, by Eq.(2.2) and a direct computation we have

$$[[f, g](x), y, z] + [x, [f', g'](y), z] + [x, y, [f'', g''](z)] = [f''', g''']([x, y, z]), \text{ for all } x, y, z \in A,$$

then $[f, g] = gf - fg \in GDer(A)$, and $([f, g], [f', g'], [f'', g''], [f''', g''']) \in \Delta(A)$. It follows that $GDer(A)$ and $QDer(A)$ are subalgebras of $gl(A)$.

For all $f, g \in Q\Gamma(A)$ and $x, y, z \in A$, we have

$$\begin{aligned} [[f, g](x), y, z] &= [(fg - gf)(x), y, z] = [g(x), f(y), z] - [f(x), g(y), z] \\ &= [x, f(y), g(z)] - [f(x), y, g(z)] = [x, y, fg(z)] - [x, y, fg(z)] = 0. \end{aligned}$$

Since $Z(A) = 0$, we obtain $[f, g] = 0$. \square

Lemma 3.2. *Let A be a 3-Lie algebra. Then the following inclusions hold*

$$Q\Gamma(A) \subseteq GDer(A), \quad (3.1)$$

$$\Gamma(A) \subseteq QDer(A) \cap Q\Gamma(A), \quad (3.2)$$

$$[Der(A), \Gamma(A)] \subseteq \Gamma(A), \quad (3.3)$$

$$[QDer(A), Q\Gamma(A)] \subseteq Q\Gamma(A). \quad (3.4)$$

Proof For all $f \in Q\Gamma(A)$, we have

$$[f(x), y, z] + [x, \frac{-1}{2}f(y), z] + [x, y, \frac{-1}{2}f(z)] = 0,$$

it implies that $(f, \frac{-1}{2}f, \frac{-1}{2}f, 0) \in \Delta(A)$. Therefore, $f \in GDer(A)$. Similarly, by direct computations, we obtain (3.2), (3.3) and (3.4). \square

The following examples to show the relations between $Der(A)$, $QDer(A)$ and $GDer(A)$.

If A is an abelian 3-Lie algebra, then it is clear that $Der(A) = QDer(A) = GDer(A) = gl(A)$.

If A is a 3-dimensional 3-Lie algebra with $A^1 \neq 0$. Then there exists a basis $\{x_1, x_2, x_3\}$ such that the multiplication is $[x_1, x_2, x_3] = x_1$. For all linear map f of A , suppose $f(x_i) = \sum_{j=1}^3 a_{ij}x_j$, $a_{ij} \in \mathbb{F}$. Define a linear map f' of A by $f'(x_i) = \sum_{j=1}^3 b_{ij}x_j$, where $b_{ij} \in \mathbb{F}$ satisfying $b_{11} = a_{11} + a_{22} + a_{33}$, $b_{12} = b_{13} = 0$. Then $(f, f, f, f') \in \Delta(A)$, that is, $f \in QDer(A)$. By a direct computation, $Der(A) \neq gl(A)$. Therefore, $Der(A) \subsetneq QDer(A) = gl(A)$.

Let B be a 4-dimensional 3-Lie algebra with the multiplication $[x_1, x_2, x_3] = x_1$, where $\{x_1, x_2, x_3, x_4\}$ is a basis of B , then $Z(B) = \mathbb{F}x_4 \neq 0$. For all $(f, f, f, f') \in \Delta(A)$, suppose $f(x_i) = \sum_{k=1}^4 a_{ik}x_k$. Then $[x_i, x_j, f(x_4)] = [x_i, x_j, \sum_{k=1}^4 a_{4k}x_k] = 0$. We obtain $a_{41} = a_{42} = a_{43} = 0$. It follows $QDer(A) \neq gl(A)$. By a direct computation, $Der(A) \neq QDer(A)$. Therefore, $Der(A) \subsetneq QDer(A) \subsetneq gl(A)$.

Theorem 3.3. *Let A be a 3-Lie algebra over \mathbb{F} . Then*

(1) $GDer(A) = QDer(A) + Q\Gamma(A)$ (the sum of subspaces).

(2) $Q\Gamma(A)$ is an ideal of $GDer(A)$. In addition, if $Z(A) = 0$, then $Q\Gamma(A)$ is an abelian ideal of $GDer(A)$.

Proof For every $g \in GDer(A)$, and $(g, g', g'', g''') \in \Delta(A)$, we see that

$$(g, g', g'', g''') = (\frac{g+g'+g''}{3}, \frac{g+g'+g''}{3}, \frac{g+g'+g''}{3}, g''') + (\frac{2g-g'-g''}{3}, \frac{2g'-g-g''}{3}, \frac{2g''-g-g'}{3}, 0).$$

Now we prove that $\frac{g+g'+g''}{3} \in QDer(A)$, and maps $\frac{2g-g'-g''}{3}, \frac{2g'-g-g''}{3}, \frac{2g''-g-g'}{3} \in Q\Gamma(A)$.

From Eq.(2.2), we find that

$$\begin{aligned} & [\frac{g+g'+g''}{3}(x), y, z] + [x, \frac{g+g'+g''}{3}(y), z] + [x, y, \frac{g+g'+g''}{3}(z)] \\ &= \frac{1}{3}([g(x), y, z] + [x, g(y), z] + [x, y, g(z)]) + \frac{1}{3}([g'(x), y, z] + [x, g'(y), z] + [x, y, g'(z)]) \\ &+ \frac{1}{3}([g''(x), y, z] + [x, g''(y), z] + [x, y, g''(z)]) \\ &= g'''([z, y, z]). \end{aligned}$$

It follows that $\frac{g+g'+g''}{3} \in QDer(A)$.

By Eqs.(2.2) and (2.3), we have

$$\begin{aligned} & [\frac{2g-g'-g''}{3}(x), y, z] - [x, \frac{2g-g'-g''}{3}(y), z] \\ &= \frac{2}{3}[g(x), y, z] - \frac{1}{3}[g'(x), y, z] - \frac{1}{3}[g''(x), y, z] - [x, \frac{2}{3}g(y), z] + [x, \frac{1}{3}g'(y), z] + [x, \frac{1}{3}g''(y), z] \\ &= \frac{2}{3}g'''([x, y, z]) - \frac{2}{3}[x, g'(y), z] - \frac{2}{3}[x, y, g''(z)] - \frac{1}{3}g'''([x, y, z]) + \frac{1}{3}[x, g(y), z] + \frac{1}{3}[x, y, g''(z)] \\ &- \frac{1}{3}g'''([x, y, z]) + \frac{1}{3}[x, g'(y), z] + \frac{1}{3}[x, y, g(z)] - \frac{2}{3}[x, g(y), z] + \frac{1}{3}[x, g'(y), z] + \frac{1}{3}[x, g''(y), z] \\ &= -\frac{1}{3}[x, y, g''(z)] + \frac{1}{3}[x, y, g(z)] - \frac{1}{3}[x, g(y), z] + \frac{1}{3}[x, g''(y), z] \\ &= \frac{1}{3}([x, y, (g-g'')(z)] + [x, (g''-g)(y), z]), \end{aligned}$$

$$g'''([x, y, z]) = [g'(x), y, z] + [x, g(y), z] + [x, y, g''(z)],$$

$$g'''([x, y, z]) = [g'(x), y, z] + [x, g''(y), z] + [x, y, g(z)].$$

Therefore,

$$[\frac{2g-g'-g''}{3}(x), y, z] = [x, \frac{2g-g'-g''}{3}(y), z] = [x, y, \frac{2g-g'-g''}{3}(z)],$$

$$[\frac{2g'-g-g''}{3}(x), y, z] = [x, \frac{2g'-g-g''}{3}(y), z] = [x, y, \frac{2g'-g-g''}{3}(z)],$$

$$[\frac{2g''-g-g'}{3}(x), y, z] = [x, \frac{2g''-g-g'}{3}(y), z] = [x, y, \frac{2g''-g-g'}{3}(z)].$$

The result (1) follows. The result (2) follows from Lemma 3.2. \square

If A is a decomposable 3-Lie algebra, that is, $A = H \oplus K$ (a direct sum of ideals). Then we have the following result.

Theorem 3.4. *Let $A = H \oplus K$ and $Z(A) = 0$. Then for all $g \in GDer(A)$, we have $g(H) \subseteq H$, $g(K) \subseteq K$, and*

$$GDer(A) = GDer(H) \oplus GDer(K), \quad QDer(A) = QDer(H) \oplus QDer(K).$$

Proof The result follows from a direct computation. \square

At last of this section, we show that quasiderivations of a 3-Lie algebra can be embedded as derivations in a larger algebra. For this, we need to construct 3-Lie algebras by tensor product [20].

Let A be a 3-Lie algebra over \mathbb{F} and t be an indeterminant. Then quotient algebra $t\mathbb{F}[t]/(t^4)$ is a 3-dimensional commutative associative algebra. Let \widetilde{A} be the 3-Lie algebra over \mathbb{F} by the tensor product of A and $t\mathbb{F}[t]/(t^4)$, that is, $\widetilde{A} = A \otimes (t\mathbb{F}[t]/(t^4))$, and write at , at^2 and at^3 in place of $a \otimes t$, $a \otimes t^2$ and $a \otimes t^3$, respectively. Then the multiplication of \widetilde{A} is

$$[a_1t^l, a_2t^l, a_3t^l] = [a_1, a_2, a_3]t^3, \forall a_1, a_2, a_3 \in A, \quad (3.5)$$

and $[a_1t^l, a_2t^m, a_3t^n] = 0$ in the case that at least one of l, m, n is larger than 1. Let U be a subspace of A satisfying $A = U \oplus A^1$, that is, the subspace U is the complement of the derived algebra $A^1 = [A, A, A]$, then

$$\widetilde{A} = At \oplus At^2 \oplus At^3 = At \oplus At^2 \oplus A^1t^3 \oplus Ut^3. \quad (3.6)$$

Define linear map $l_u : QDer(A) \mapsto Hom(\widetilde{A}, \widetilde{A})$, by for all $(f, f, f, f') \in \Delta(A)$, and $a, b \in A, c \in A^1, u \in U$,

$$l_u(f)(at + bt^2 + ct^3 + ut^3) = f(a)t + f'(c)t^3. \quad (3.7)$$

Then l_u is injective, and $l_u(f)$ does not depend on the choice of f' .

Theorem 3.5. *Let A be a 3-Lie algebra, \widetilde{A} and l_u be defined as Eqs.(3.6) and (3.7), respectively. Then $l_u(QDer(A)) \subseteq Der(\widetilde{A})$. In addition, if $Z(A) = 0$, then $Der(\widetilde{A})$ has a semidirect summation $Der(\widetilde{A}) = l_u(QDer(A)) \oplus Z(Der(\widetilde{A}))$.*

Proof For all $a_it + b_it^2 + c_it^3 + u_it^3 \in \widetilde{A}, i = 1, 2, 3$, by Eqs.(3.6) and (3.7),

$$\begin{aligned} & [l_u(f)(a_1t + b_1t^2 + c_1t^3 + u_1t^3), a_2t + b_2t^2 + c_2t^3 + u_2t^3, a_3t + b_3t^2 + c_3t^3 + u_3t^3] \\ & + [a_1t + b_1t^2 + c_1t^3 + u_1t^3, l_u(f)(a_2t + b_2t^2 + c_2t^3 + u_2t^3), a_3t + b_3t^2 + c_3t^3 + u_3t^3] \\ & + [a_1t + b_1t^2 + c_1t^3 + u_1t^3, a_2t + b_2t^2 + c_2t^3 + u_2t^3, l_u(f)(a_3t + b_3t^2 + c_3t^3 + u_3t^3)] \\ & = [f(a_1), a_2, a_3]t^3 + [a_1, f(a_2), a_3]t^3 + [a_1, a_2, f(a_3)]t^3 \\ & = f'([a_1, a_2, a_3])t^3, \end{aligned}$$

$$\begin{aligned} & l_u(f)([a_1t + b_1t^2 + c_1t^3 + u_1t^3, a_2t + b_2t^2 + c_2t^3 + u_2t^3, a_3t + b_3t^2 + c_3t^3 + u_3t^3]) \\ & = l_u(f)([a_1, a_2, a_3]t^3) = f'([a_1, a_2, a_3])t^3. \end{aligned}$$

Therefore, $l_u(f) \in Der(\widetilde{A})$. It follows that $l_u(QDer(A)) \subseteq Der(\widetilde{A})$.

Now if $Z(A) = 0$, then by Eq.(3.5) the center of 3-Lie algebra \widetilde{A} is

$$Z(\widetilde{A}) = At^2 + At^3, \quad (3.8)$$

and for all $D \in \text{Der}(\widetilde{A})$, $D(Z(\widetilde{A})) \subseteq Z(\widetilde{A})$. Then for all linear map

$$f : At \oplus At^2 \oplus Ut^3 \mapsto At^2 \oplus At^3,$$

f can be extended to an element of $Z(\text{Der}(\widetilde{A}))$ by taking $f(A^1t^3) = 0$.

For every $g \in \text{Der}(\widetilde{A})$ and $a \in A$, suppose

$$g(at) = a't + b't^2 + c't^3 + u't^3, \text{ where } a', b' \in A, c' \in A^1, u' \in U. \quad (3.9)$$

Define linear map $f : \widetilde{A} \rightarrow \widetilde{A}$ by for all $a, b \in A, c \in A^1, u \in U$,

$$f(at) = b't^2 + c't^3 + u't^3, \quad f(ct^3) = 0, \quad f(bt^2 + ut^3) = g(bt^2 + ut^3).$$

By Eq.(3.8), $f \in Z(\text{Der}(\widetilde{A}))$, and

$$(g - f)(at) = a't, \quad (g - f)(ct^3) = c't^3.$$

From Eq.(3.9), there exist linear maps h, h' of A satisfying for all $a \in A, c \in A^1, u \in U$,

$$h(a) = a', \quad h'(c) = c', \quad h'(u) = 0,$$

then $(h, h, h, h') \in \Delta(A)$, that is, $h \in Q\text{Der}(A)$. Thanks to Eq.(3.7),

$$g - f = l_u(h).$$

The proof is completed. \square

4. QUASIDERIVATIONS OF 3-LIE ALGEBRAS

In this section, we first describe quasiderivations of 3-Lie algebras by cohomology theory, and then we study quasiderivation algebras of a class of 3-Lie algebras which contains a maximal diagonalized tours T . For convenience, for a 3-Lie algebra A and for all $x, y, z \in A$, we denote the multiplication $[x, y, z]$ by $\mu(x, y, z)$.

Let (A, μ) be a 3-Lie algebra. Denote the kernel of the multiplication μ by

$$\text{Ker}(\mu) = \{v \otimes w \otimes u \mid \mu(v, w, u) = 0, v, w, u \in A\}. \quad (4.1)$$

For every linear map f of A , define a linear map $f^* : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ by for all $v, w, u \in A$,

$$f^*(v \otimes w \otimes u) = f(v) \otimes w \otimes u + v \otimes f(w) \otimes u + v \otimes w \otimes f(u). \quad (4.2)$$

Theorem 4.1 *Let (A, μ) be a 3-Lie algebra and $f \in \text{Hom}(A, A)$. Then $f \in Q\text{Der}(A, \mu)$ if and only if*

$$f^*(\text{Ker}(\mu)) \subseteq \text{Ker}(\mu), \quad (4.3)$$

that is, for all $v, w, u \in A$, if $\mu(v, w, u) = 0$, then $\mu(f(v), w, u) + \mu(v, f(w), u) + \mu(v, w, f(u)) = 0$.

Proof For $f \in QDer(A)$ and $(f, f, f, f') \in \Delta(A)$, by Eqs.(4.1) and (4.2), for all $v \otimes w \otimes u \in Ker(\mu)$,

$$\mu(f^*(v \otimes w \otimes u)) = \mu(f(v), w, u) + \mu(v, f(w), u) + \mu(v, w, f(u)) = f'(\mu(v, w, u)) = 0.$$

Then $f^*(Ker(\mu)) \subseteq Ker(\mu)$.

Conversely, let U be a complement of the derived algebra A^1 in A , that is, $A = A^1 \oplus U$. For $f \in Hom(A, A)$, if f satisfies inclusion (4.3). Define linear map $f' : A \rightarrow A$ by for all $u \in U$, $f'(u) = 0$, and for every $z = \sum_{i=1}^m \mu(v_i, w_i, u_i) \in A^1$,

$$f'(z) = \sum_{i=1}^m \mu(f^*(v_i \otimes w_i \otimes u_i)).$$

Then f' is well defined.

In fact, if $z = \sum_{i=1}^m \mu(v_i, w_i, u_i)$ and $z = \sum_{j=1}^l \mu(v'_j, w'_j, u'_j)$, where $v_i, w_i, u_i, v'_i, w'_i, u'_i \in A$, $1 \leq i \leq m$, $1 \leq j \leq l$, then

$$\sum_{i=1}^m (v_i \otimes w_i \otimes u_i) - \sum_{j=1}^l (v'_j \otimes w'_j \otimes u'_j) \in Ker\mu.$$

Thanks to inclusion (4.3), $\mu(f^*(\sum_{i=1}^m (v_i \otimes w_i \otimes u_i))) - \mu(f^*(\sum_{j=1}^l (v'_j \otimes w'_j \otimes u'_j))) = 0$.

Therefore, for all $v, w, u \in A$,

$$\mu(f(v), w, u) + \mu(v, f(w), u) + \mu(v, w, f(u)) = \mu(f^*(v \otimes w \otimes u)) = f'(\mu(v, w, u)).$$

It follows $f \in QDer(A)$. \square

In papers [22, 23, 24], authors introduced modules and cohomology theory of 3-Lie algebras. For describing quasiderivations, we first recall some definitions. Let A be a 3-Lie algebra, V be a vector space. If there exists a linear map $\alpha : A \wedge A \rightarrow End(V)$ satisfying

$$\begin{aligned} \alpha([x, y, z], w) &= \alpha(y, z)\alpha(x, w) + \alpha(z, x)\alpha(y, w) + \alpha(x, y)\alpha(z, w), \\ 0 &= \alpha(z, w)\alpha(x, y) - \alpha(x, y)\alpha(z, w) + \alpha([x, y, z], w) + \alpha(z, [x, y, w]) \end{aligned}$$

for all $x, y, z, w \in A$, then (V, α) is called a representation of A , or V is an A -module. If $\alpha(x, y) = 0$ for all $x, y \in A$, then V is a trivial module. If $V = A$ and $\alpha(x, y) = \text{ad}(x, y)$ for all $x, y \in A$, then A is called the adjoin module of A .

A 3-Lie algebra A as an adjoint module, we define

$$\begin{aligned} C^0(A, A) &= Hom(A, A), \quad C^1(A, A) = Hom(A \otimes A \otimes A, A), \quad C^2(A, A) = H(A^{\otimes 5}, A), \\ \text{and } \delta_0 : C^0(A, A) &\rightarrow C^1(A, A), \quad \delta_1 : C^1(A, A) \rightarrow C^2(A, A) : \text{for all } x_i \in A, 1 \leq i \leq 5, \end{aligned}$$

$$\delta_0(f)(x_1, x_2, x_3) = f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)], \quad (4.4)$$

$$\begin{aligned}\delta_1(f)(x_1, x_2, x_3, x_4, x_5) &= \sum_{i=1}^3 f(x_1, \dots, [x_i, x_4, x_5], \dots, x_3) - f([x_1, x_2, x_3], x_4, x_5) \\ &\quad + \sum_{i=1}^3 ([x_1, \dots, f(x_i, x_4, x_5), \dots, x_3]) - [f(x_1, x_2, x_3), x_4, x_5].\end{aligned}\quad (4.5)$$

Denote $\text{Ker}\delta_i = Z^i(A, A) \subseteq C^i(A, A)$, $B^{i+1}(A, A) = \text{Im}(\delta_i) \subseteq C^{i+1}(A, A)$, $i = 0, 1$. Thanks to Eqs.(4.4) and (4.5), $\delta_1\delta_0 = 0$.

Let \dot{A} be the trivial module of A on the underlying vector space of A , and

$$C^0(A, \dot{A}) = \text{Hom}(A, \dot{A}), C^1(A, \dot{A}) = \text{Hom}(A \otimes A \otimes A, \dot{A}), C^2(A, \dot{A}) = H(A^{\otimes 5}, \dot{A}).$$

Define $\dot{\delta}_0 : C^0(A, \dot{A}) \rightarrow C^1(A, \dot{A})$, $\dot{\delta}_1 : C^1(A, \dot{A}) \rightarrow C^2(A, \dot{A})$ by for all $x_i \in A$, $1 \leq i \leq 5$,

$$\dot{\delta}_0(f)(x, y, z) = f([x, y, z]), \quad (4.6)$$

$$\dot{\delta}_1(f)(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^3 f(x_1, [x_i, x_4, x_5], x_3) - f([x_1, x_2, x_3], x_4, x_5), \quad (4.7)$$

then $\dot{\delta}_1\dot{\delta}_0 = 0$, and have the following result.

Theorem 4.2 *Let A be a 3-Lie algebra, f, f' be linear maps of A . Then*

1) $f \in \text{QDer}(A)$ if and only if $\delta_0(f) \in B^1(A, \dot{A})$. More specifically, $(f, f, f, f') \in \Delta(A)$ if and only if $\delta_0(f) = \dot{\delta}_0(f - f')$.

2) If $\dot{\delta}_0(f) \in Z^1(A, A)$, then for all $x, y, z, u, v \in A$,

$$[f([x, y, z]), u, v] = [f([x, u, v]), y, z] + [x, f([y, u, v]), z] + [x, y, f([z, u, v])]. \quad (4.8)$$

In particular, if $(f, f, f, f') \in \Delta(A)$, then

$$[(f - f')([x, y, z]), u, v] = [(f - f')([x, u, v]), y, z] + [x, (f - f')([y, u, v]), z] + [x, y, (f - f')([z, u, v])].$$

Proof If $f \in \text{QDer}(A)$ and $(f, f, f, f') \in \Delta(A)$, by Eq.(4.4), we find that

$$\delta_0(f)(x, y, z) = (f - f')([x, y, z]) = \dot{\delta}_0(f - f')(x, y, z).$$

Therefore, $\delta_0(f) \in B^1(A, \dot{A})$, and $\delta_0 f = \dot{\delta}_0(f - f')$.

Conversely, if $\delta_0(f) \in B^1(A, \dot{A})$, then there exists $f' \in C^0(A, \dot{A})$ such that $\delta_0(f) = \dot{\delta}_0 f'$, that is, for all $x, y, z \in A$,

$$\delta_0(f)(x, y, z) = f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] = \delta_0 f'([x, y, z]) = f'([x, y, z]).$$

It follows that $(f' - f)([x, y, z]) = [f(x), y, z] + [x, f(y), z] + [x, y, f(z)]$. The result 1) holds.

If $f \in \text{Hom}(A, A)$ satisfies $\dot{\delta}_0 f \in Z^1(A, A)$, then $\delta_1\dot{\delta}_0(f) = 0$. Thanks to Eq.(4.5), for all $x, y, z, u, v \in A$,

$$\begin{aligned}\delta_1\dot{\delta}_0(f)(x, y, z, u, v) &= [\dot{\delta}_0(f)(x, u, v), y, z] + [x, \dot{\delta}_0(f)(y, u, v), z] + [x, y, \dot{\delta}_0(f)(z, u, v)] - [\dot{\delta}_0(f)(x, y, z), u, v] \\ &= [f([x, u, v]), y, z] + [x, f([y, u, v]), z] + [x, y, f([z, u, v])] - [f([x, y, z]), u, v] = 0.\end{aligned}$$

It follows Eq.(4.8).

If $(f, f, f, f') \in \Delta(A)$, then $f \in QDer(A)$. Thanks to result 1), $\delta_0(f) = \dot{\delta}_0(f - f')$. From $\dot{\delta}_0(f) \in Z^1(A, A)$ and $\delta_0(f) \in Z^1(A, A)$, we obtain that $f - f'$ satisfies Eq.(4.8). \square

In the following of this section, we study structures of qusiderivation algebras of a class of 3-Lie algebras which containing a maximal diagonalized tours.

Let A be a 3-Lie algebra over an algebraically closed field \mathbb{F} with $chF = 0$, which contains a maximal diagonalized tours T , that is, T is an abelian subalgebra, and A has a root-subspace decomposition associative T as follows

$$A = \sum_{\gamma \in \Omega} A_\gamma, \quad A_\gamma = \{x \in A \mid \text{for all } t_1, t_2 \in T, \text{ad}(t_1, t_2)(x) = \gamma(t_1, t_2)x, \}, \quad A_0 = T, \quad (4.9)$$

where $\Omega \subseteq (T \wedge T)^* - \{0\}$ - the dual space of $T \wedge T$. If $A_\alpha \neq 0$, α is called a weight of A associative T .

For example, let A be a 4-dimensional 3-Lie algebra over the complex field \mathbb{F} (see Lemm 3.1 in [21]). Then except cases (b^1) and (c^2) , the subalgebra spanned by e_3, e_4 is a maximal diagonalized tours of A .

Now let A be a 3-Lie algebra containing a maximal diagonalized tours T . For all $t_1, t_2 \in T$, we define linear mapping $d(t_1, t_2) : Hom(A, A) \rightarrow Hom(A, A)$ by for all $f \in Hom(A, A)$ and $x \in A$,

$$(d(t_1, t_2)f)(x) = [t_1, t_2, f(x)] - f([t_1, t_2, x]) = (\text{ad}(t_1, t_2)f - f\text{ad}(t_1, t_2))(x). \quad (4.10)$$

For convenient, sometimes we replace $d(t_1, t_2)f$ by $(t_1, t_2)f$, for all $t_1, t_2 \in T$, $f \in Hom(A, A)$.

In the following of this section, let A be a 3-Lie algebra over an algebraically closed field \mathbb{F} with $ch\mathbb{F} = 0$, which contains a maximal diagonalized tours T , and $A_0 = T$.

Lemma 4.3 *Let A be a 3-Lie algebra over \mathbb{F} . Then for all $t_1, t_2, t_3, t_4 \in T$ and $f \in QDer(A)$,*

- 1) $(t_1, t_2)f \in QDer(A)$, that is, $(T, T)QDer(A) \subseteq QDer(A)$;
- 2) $(t_1, t_2)(t_3, t_4)f - (t_3, t_4)(t_1, t_2)f = 0$;
- 3) $((t_1, t_2)^n \cdot f)(x) = \sum_{k=0}^n (-1)^{k+1} C_n^k \text{ad}_n^{k-m-k}(t_1, t_2) f \text{ad}^k(t_1, t_2)(x).$

(4.11)

Proof For arbitrary $f \in QDer(A)$, $t_1, t_2 \in T$ and $x, y, z \in A$, suppose $(f, f, f, f') \in \Delta(A)$, by Eq.(4.10), we have

$$\begin{aligned}
& [(t_1, t_2)f(x), y, z] + [x, y, (t_1, t_2)f(y), z] + [x, y, (t_1, t_2)f(z)] \\
&= [\text{ad}(t_1, t_2)f(x), y, z] + [f(x), \text{ad}(t_1, t_2)(y), z] + [f(x), y, \text{ad}(t_1, t_2)(z)] \\
&+ [\text{ad}(t_1, t_2)(x), f(y), z] + [x, \text{ad}(t_1, t_2)f(y), z] + [x, f(y), \text{ad}(t_1, t_2)(z)] \\
&+ [\text{ad}(t_1, t_2)(x), y, f(z)] + [x, \text{ad}(t_1, t_2)(y), f(z)] + [x, y, \text{ad}(t_1, t_2)f(z)] \\
&- ([f(\text{ad}(t_1, t_2)(x)), y, z] + [\text{ad}(t_1, t_2)(x), f(y), z] + [\text{ad}(t_1, t_2)(x), y, f(z)]) \\
&- ([f(x), \text{ad}(t_1, t_2)(y), z] + [x, f(\text{ad}(t_1, t_2)(y)), z] + [x, \text{ad}(t_1, t_2)(y), f(z)]) \\
&- ([f(x), y, \text{ad}(t_1, t_2)(z)] + [x, f(y), \text{ad}(t_1, t_2)(z)] + [x, y, f(\text{ad}(t_1, t_2)(z))]) \\
&= \text{ad}(t_1, t_2)f'([x, y, z]) - f'\text{ad}(t_1, t_2)([x, y, z]).
\end{aligned}$$

It follows that $((t_1, t_2)f, (t_1, t_2)f, (t_1, t_2)f, \text{ad}(t_1, t_2)f' - f'\text{ad}(t_1, t_2)) \in \Delta(A)$. The result 1) follows.

The result 2) follows from Eq.(4.10) and a direct computation. Now we prove Eq.(4.11).

In the case $n = 2$, we have

$$\begin{aligned}
((t_1, t_2)^2 f)(x) &= [t_1, t_2, ((t_1, t_2)f)(x)] - ((t_1, t_2)f)([t_1, t_2, x]) \\
&= [t_1, t_2, [t_1, t_2, f(x)] - f([t_1, t_2, x])] \\
&- [t_1, t_2, f([t_1, t_2, x])] + f([t_1, t_2, [t_1, t_2, x]]) \\
&= \text{ad}^2(t_1, t_2)f(x) - 2\text{ad}(t_1, t_2)f\text{ad}(t_1, t_2)(x) + f\text{ad}^2(t_1, t_2)(x).
\end{aligned}$$

Suppose identity (4.11) is true for the case n , then

$$\begin{aligned}
((t_1, t_2)^{n+1} f)(x) &= ((t_1, t_2)((t_1, t_2)^n f))(x) \\
&= ((t_1, t_2)(\sum_{k=0}^n (-1)^{k+1} C_n^k \text{ad}^{n-k}(t_1, t_2)f \text{ad}^k(t_1, t_2)))(x) \\
&= (\sum_{k=0}^n (-1)^{k+1} C_n^k \text{ad}^{n+1-k}(t_1, t_2)f \text{ad}^k(t_1, t_2))(x) \\
&+ (\sum_{k=0}^n (-1)^{k+2} C_n^k \text{ad}^{n-k}(t_1, t_2)f \text{ad}^{k+1}(t_1, t_2))(x) \\
&= \sum_{k=0}^n (-1)^{k+1} C_n^k \text{ad}^{n+1-k}(t_1, t_2)f \text{ad}^k(t_1, t_2)(x) \\
&+ \sum_{l=1}^{n+1} (-1)^{l+1} C_{n+1}^l \text{ad}^{n+1-l}(t_1, t_2)f \text{ad}^l(t_1, t_2)(x) \\
&= \sum_{k=0}^{n+1} (-1)^{k+1} C_{n+1}^k \text{ad}^{n+1-k}(t_1, t_2)f \text{ad}^k(t_1, t_2)(x).
\end{aligned}$$

The proof is completed. \square

From Lemma 4.3, we can assume the Fitting decomposition of $QDer(A)$ associative T as

$$QDer(A) = QDer(A)_0 \bigoplus \sum_{\alpha \in \Lambda} QDer(A)_\alpha, \quad (4.12)$$

where $\Lambda \subseteq (T \wedge T)^* - \{0\}$, and

$QDer(A)_0 = \{f \mid f \in QDer(A), \text{ for all } t_1, t_2 \in T, \text{ there exists a positive integer } m \text{ such that } (t_1, t_2)^m f = 0\}$,

$QDer(A)_\alpha = \{f \mid f \in QDer(A), \text{ for all } t_1, t_2 \in T, \text{ there exists a positive integer } m \text{ such that } ((t_1, t_2) - \alpha(t_1, t_2)I_d)^m f = 0\}$.

For $\alpha \in \Lambda$, if $QDer(A)_\alpha \neq 0$, then α is called a weight of T .

Lemma 4.4 *Let A be a 3-Lie algebra. Then for all $\alpha \in \Lambda$ and $\gamma \in \Omega$,*

$$QDer(A)_\alpha(A_\gamma) \subseteq A_{\alpha+\gamma}.$$

Proof For all $f \in QDer(A)_\alpha$, $x_\gamma \in A_\gamma$ and $t_1, t_2 \in T$, suppose $f(x_\gamma) = \sum_{\beta \in (T \wedge T)^*} x_\beta$, where $x_\beta \in A_\beta$. Then by Eqs.(4.9)-(4.11) and Lemma 4.3, there exists an integer m such that

$$((t_1, t_2) - \alpha(t_1, t_2)I_d)^m f = 0.$$

Since

$$\begin{aligned} & (((t_1, t_2) - \alpha(t_1, t_2)I_d)^m f)(x_\gamma) \\ &= \sum_{s=0}^m (-1)^{(m-s)} C_m^s \alpha(t_1, t_2)^{m-s} ((t_1, t_2)^s f)(x_\gamma) \\ &= \sum_{s=0}^m (-1)^{(m-s)} C_m^s \alpha(t_1, t_2)^{m-s} \sum_{k=0}^s (-1)^{k+1} C_s^k \gamma(t_1, t_2)^k ad(t_1, t_2)^{s-k} f(x_\gamma) \\ &= \sum_{\beta \in (T \wedge T)^*} \sum_{s=0}^m (-1)^{(m-s)} C_m^s \alpha(t_1, t_2)^{m-s} \sum_{k=0}^s (-1)^{k+1} C_s^k \gamma(t_1, t_2)^k \beta(t_1, t_2)^{s-k} x_\beta \\ &= \sum_{\beta \in (T \wedge T)^*} \sum_{s=0}^m (-1)^{(m-s-1)} C_m^s \alpha(t_1, t_2)^{m-s} (\beta - \gamma)(t_1, t_2)^s x_\beta \\ &= - \sum_{\beta \in (T \wedge T)^*} (\beta - \gamma - \alpha)(t_1, t_2) x_\beta \\ &= (\gamma + \alpha)(t_1, t_2) f(x_\gamma) - (t_1, t_2) f(x_\gamma), \end{aligned}$$

we have $f(x_\gamma) \in A_{\alpha+\gamma}$. \square

Theorem 4.5 *Let A be a 3-Lie algebra. Then the action of T on $QDer(A)$ defined by Eq.(4.10) is diagonally, that is,*

$$QDer(A)_\alpha = \{f \mid f \in QDer(A), \text{ for all } t_1, t_2 \in T, (t_1, t_2)f = \alpha(t_1, t_2)f\}. \quad (4.13)$$

Therefore, for all $f \in QDer(A)_0$ and $t_1, t_2 \in T$, $(t_1, t_2)f = 0$. And if $f \in QDer(A)_0 \cap Der(A)$, then for all $t_1, t_2 \in T$ and $\gamma \in \Omega$, $\gamma(f(t_1), t_2) + \gamma(t_1, f(t_2)) = 0$.

Proof By Eq.(4.10) and Lemma 4.4, for all $f \in QDer(A)_\alpha$, $x_\gamma \in A_\gamma$ and $t_1, t_2 \in T$,

$$\begin{aligned} & ((t_1, t_2)f)(x_\gamma) \\ &= [t_1, t_2, f(x_\gamma)] - f([t_1, t_2, x_\gamma]) \\ &= (\alpha + \gamma)(t_1, t_2)f(x_\gamma) - \gamma(t_1, t_2)f(x_\gamma) = \alpha(t_1, t_2)f(x_\gamma). \end{aligned}$$

The identity (4.13) holds.

If $f \in Der(A) \cap QDer(A)_0$, then for all $\gamma \in \Omega$, $x_\gamma \in A_\gamma$ and $x_\gamma \neq 0$, by Lemma 4.4

$$\begin{aligned} & [f(t_1), t_2, x_\gamma] + [t_1, f(t_2), x_\gamma] + [t_1, t_2, f(x_\gamma)] \\ &= (\gamma(f(t_1), t_2) + \gamma(t_1, f(t_2)))x_\gamma + \gamma(t_1, t_2)f(x_\gamma) \\ &= \gamma(t_1, t_2)f(x_\gamma). \end{aligned}$$

Therefore, $\gamma(f(t_1), t_2) + \gamma(t_1, f(t_2)) = 0$. The result follows. \square

At last of this section, we give an example to show that there does not exist inherent relations between $QDer(A)_\alpha$ and the derivation algebra $Der(A)$.

Let A be a 3-Lie algebra with a basis x_1, x_2, x_3 , and the multiplication is $[x_1, x_2, x_3] = x_1$. Then $T = \mathbb{F}x_2 + \mathbb{F}x_3$ is a maximal diagonalized tours, and $A = T \dot{+} A_1$, where $A_1 = \mathbb{F}x_1$. For all $f \in Hom(A, A)$, suppose $f(x_i) = \sum_{j=1}^3 a_{ij}x_j$, $1 \leq i \leq 3$. Then the matrix form of f in the basis is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, a_{ij} \in \mathbb{F}, 1 \leq i, j \leq 3. \text{ By a direct computation, we have}$$

1). $QDer(A) = gl(A)$, and for all $f \in gl(A)$, $(f, f, f, f') \in \Delta(A)$ if and only if

$$f' = \begin{pmatrix} a_{11} + a_{22} + a_{33} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, b_{ij} \in \mathbb{F}, 1 \leq i, j \leq 3.$$

2). $QDer(A) = QDerA_0 \oplus QDerA_1 \oplus QDerA_{-1}$, where

$$QDer(A)_0 = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \mid a_{ij} \in \mathbb{F}, 1 \leq i, j \leq 3 \right\}.$$

$$QDer(A)_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \mid a_{j1} \in \mathbb{F}, j = 2, 3 \right\}.$$

$$QDer(A)_1 = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a_{1j} \in \mathbb{F}, j = 2, 3 \right\}.$$

$$3). Der(A) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{F}, 1 \leq i, j \leq 3 \right\}.$$

$$ad(A) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \mid a_i \in \mathbb{F}, 1 \leq i \leq 3 \right\}.$$

Then $QDer(A)_{-1} \subseteq Der(A)$, $QDer(A)_1 \not\subseteq Der(A)$, $QDer(A)_0 \not\subseteq Der(A)$, $Der(A) \not\subseteq QDer(A)_0$.

5. QUASICENTROIDS OF 3-LIE ALGEBRAS

From Theorem 3.3, general derivation algebra of a 3-Lie algebra is generated by quasiderivations and quasicentroid. In this section, we study the structure of quasicentroid of 3-Lie algebras. From section 3, the centroid $\Gamma(A)$ is contained in $QDer(A)$, and $\Gamma(A)$, $Q\Gamma(A)$ are associative algebras. In addition, for the centerless 3-Lie algebra A , $\Gamma(A)$ and $Q\Gamma(A)$ are commutative.

Lemma 5.1 *Let A be a 3-Lie algebra, then*

- 1) $[\Gamma(A), Q\Gamma(A)] \subseteq Hom(A, Z(A))$;
- 2) if $f \in \Gamma(A)$, then $Ker(f)$ and $Im(f)$ are ideals of A ;
- 3) if A is indecomposable and for $f \in \Gamma(A)$ satisfying that x^2 does not divide the minimal polynomial of f , then f is invertible;
- 4) if A is indecomposable and the centroid $\Gamma(A)$ consists of semisimple elements, then $\Gamma(A)$ is a field.

Proof The result follows from a direct computation. \square

From Lemma 3.2, quasicentroid $Q\Gamma(A)$ is a 3-Lie algebra A -module under the action defined by for all $f \in Q\Gamma(A)$ and $x, y, z \in A$,

$$((x, y)f)(z) = (ad(x, y)f - fad(x, y))(z). \quad (5.1)$$

Therefore $Q\Gamma(A)$ is a T -module, and we have the following identities.

Lemma 5.2 Let A be a 3-Lie algebra, and $ch\mathbb{F} \neq 2$. Then for all $f \in Q\Gamma(A)$ and $x, y \in A$,

$$\begin{aligned} [x, f(x), y] &= 0, \\ \text{ad}(x, y)\text{ad}(f(x), y) &= \text{ad}(f(x), y)\text{ad}(x, y), \\ ((x, y)f)(z) &= ((y, z)f)(x) = ((z, x)f)(y), \\ \text{ad}^m(f(x), y) &= \text{ad}^m(x, y)f^m, \text{ for all positive integer } m, \\ \text{ad}^{m+1}(x, y)f &= \text{ad}(f(x), y)\text{ad}^m(x, y), \text{ for all positive integer } m. \end{aligned}$$

Proof For all $f \in Q\Gamma(A)$ and $x, y, z \in A$, since $[x, f(x), y] = [f(x), x, y]$ and $ch\mathbb{F} \neq 2$, we have $[x, f(x), y] = 0$.

Follows from $[\text{ad}(x, y), \text{ad}(f(x), y)] = \text{ad}([x, y, f(x)], y) + \text{ad}(f(x), [x, y, y]) = 0$, that $\text{ad}(x, y)$ and $\text{ad}(f(x), y)$ are commutative.

The identity $((x, y)f)(z) = ((y, z)f)(x) = ((z, x)f)(y)$ follows from the skew-symmetry of the multiplication of 3-Lie algebras, directly.

For the case $m = 2$

$$\begin{aligned} \text{ad}^2(f(x), y)(z) &= \text{ad}(f(x), y)\text{ad}(f(x), y)(z) = [f(x), y, [f(x), y, z]] = [f(x), y[x, y, f(z)]] \\ &= [[f(x), y, x], y, f(z)] + [x, [f(x), y, y], f(z)] + [x, y, [f(x), y, f(z)]] \\ &= \text{ad}^2(x, y)f^2(z). \end{aligned}$$

Then for the case m

$$\begin{aligned} \text{ad}^m(f(x), y)(z) &= \text{ad}(f(x), y)\text{ad}^{m-1}(f(x), y)(z) = \text{ad}(f(x), y)\text{ad}^{m-1}(x, y)f^{m-1}(z) \\ &= \text{ad}^{m-1}(x, y)\text{ad}(f(x), y)f^{m-1}(z) = \text{ad}^{m-1}(x, y)[f(x), y, f^{m-1}(z)] \\ &= \text{ad}^{m-1}(x, y)([x, y, f^m])(z) = \text{ad}^m(x, y)f^m(z). \end{aligned}$$

Similar discussion, the last identity holds. \square

Let A be a 3-Lie algebra over an algebraically closed field \mathbb{F} of characteristic zero, which contains a maximal diagonalized tours T . By the discussion in section 4, A has decomposition (4.9) associative T . And $Q\Gamma(A)$ as a T -module in Eq.(5.1) has Fitting decomposition

$$Q\Gamma(A) = Q\Gamma(A)_0 + Q\Gamma(A)_1, \text{ where } Q\Gamma(A)_1 = \sum_{\alpha \in \Pi} Q\Gamma(A)_\alpha, \quad (5.2)$$

where $\Pi \subseteq (T \wedge T)^* - \{0\}$, and $Q\Gamma(A)_\alpha = \{f \in Q\Gamma(A) \mid \text{for all } t_1, t_2 \in T, (t_1, t_2)f = ((t_1, t_2) - \alpha(t_1, t_2)I_d)^m f = 0, \text{ for some positive integer } m\}$.

In the following we suppose that A is a 3-Lie algebra over an algebraically closed field \mathbb{F} with $ch\mathbb{F} = 0$, which contains a maximal diagonalized tours T with $A_0 = T$.

By the completely similar discussion to section 4, we have the following result.

Theorem 5.3 *Let A be a 3-Lie algebra. Then*

1) as a T -module, the decomposition (5.2) of $Q\Gamma(A)$ is diagonally, that is,

$$Q\Gamma(A)_\alpha = \{f \mid f \in Q\Gamma(A), (t_1, t_2)f = \alpha(t_1, t_2)f, \text{ for all } t_1, t_2 \in T\}; \quad (5.3)$$

2) for all $\alpha \in \Pi$ and $\gamma \in \Omega$, $Q\Gamma(A)_\alpha(A_\gamma) \subseteq A_{\alpha+\gamma}$. Therefore, $Q\Gamma(A)(T) \subseteq T$.

Proof The result 1) follows from the similar discussion to Lemma 4.4 and Theorem 4.5.

For all $f \in Q\Gamma(A)$ and $x, y, z \in T$, by Lemma 5.2

$$\text{ad}^{m+1}(x, y)f(z) = \text{ad}^m(x, y)\text{ad}(f(x), y)(z) = \text{ad}(f(x), y)\text{ad}^m(x, y)(z) = 0.$$

Therefore, $f(T) \subseteq T$. \square

Theorem 5.4 *Let A be a 3-Lie algebra. Then*

1) $Q\Gamma(A)_0(A_1) \subseteq A_1$.

2) $Q\Gamma(A)_1(T) = 0$.

3) $Q\Gamma(A)_1(A_1) \subseteq Z_A(T) = \{x \in A \mid [x, T, A] = 0\}$, therefore, $Q\Gamma(A)_1(A) \subseteq Z_A(T)$.

4) For $\alpha, \beta \in \Omega$, if $\alpha + \beta \neq 0$, then $(A_\alpha, A_\beta)Q\Gamma(A)_0 = 0$, and

$$(A_\alpha, A_{-\alpha})Q\Gamma(A)_0 \subseteq Q\Gamma(A)_0, (A_\alpha, A_{-\alpha})Q\Gamma(A)_0(A_1) = 0.$$

5) $((T, A_1)Q\Gamma(A)_0)(T) = 0$.

Proof Define linear map $\sigma : Q\Gamma(A) \otimes A \rightarrow A$ by for all $f \in Q\Gamma(A)$ and $z \in A$,

$$\sigma(f \otimes z) = f(z).$$

Thanks to Eq.(5.1), $\sigma((x, y)f \otimes z + f \otimes \text{ad}(x, y)z) = \text{ad}(x, y)\sigma(f \otimes z)$, that is, σ is a module homomorphism. Therefore, $Q\Gamma(A)_0(A_1) \subseteq A_1$. The result 1) holds.

Thanks to Theorem 5.3, $Q\Gamma(A)_1(T) \subseteq T$. But from result 1), $Q\Gamma(A)_1(T) \subseteq A_1$. Therefore, $Q\Gamma(A)_1(T) \subseteq A_1 \cap T = 0$. The result 2) holds.

The result 3) follows from the result 2) and a simple computation.

For all $\alpha, \beta \in \Omega$, $x_\alpha \in A_\alpha, y_\beta \in A_\beta, z \in A$ and $h_1, h_2 \in T, f_0 \in Q\Gamma(A)_0$,

If $\alpha + \beta \neq 0$, then

$$\begin{aligned} & ((h_1, h_2)(x_\alpha, x_\beta)f_0)(z) \\ &= -((x_\alpha, x_\beta)(h_1, h_2)f_0)(z) + ([(h_1, h_2), (x_\alpha, x_\beta)]f_0)(z) \\ &= (\alpha + \beta)(h_1, h_2)((x_\alpha, x_\beta)f_0)(z), \text{ we get} \end{aligned}$$

$$(x_\alpha, x_\beta)f_0 \in Q\Gamma(A)_{\alpha+\beta} \subseteq Q\Gamma(L)_1. \quad (5.4)$$

For all $\delta \in \Omega$, by Theorem 5.3 and the result 3),

$$(x_\alpha, x_\beta)Q\Gamma(A)_0(A_\delta) \subseteq Q\Gamma(A)_{\alpha+\beta}(A_\delta) \subseteq A_{\alpha+\beta+\delta}, \quad (5.5)$$

and $(x_\alpha, x_\beta)Q\Gamma(A)_0(A_\delta) \subseteq T$. We find that if $\alpha + \beta + \delta \neq 0$, then $(x_\alpha, x_\beta)Q\Gamma(A)_0(A_\delta) = 0$.

Now suppose $((x_\alpha, x_\beta)Q\Gamma(A)_0)(A_1) \neq 0$. then there exist $\delta, \lambda, \mu \in \Omega$, and nonzero vectors $z_\delta \in A_\delta, m_\lambda \in A_\lambda, n_\mu \in A_\mu$, and $f \in Q\Gamma(A)_0$ such that

$$[((x_\alpha, y_\beta)f)(z_\delta), m_\lambda, n_\mu] \neq 0.$$

Then

$$[((x_\alpha, x_\beta)f)(z_\delta), m_\lambda, n_\mu] = [z_\delta, ((x_\alpha, x_\beta)f)(m_\lambda), n_\mu] = [z_\delta, m_\lambda, ((x_\alpha, x_\beta)f)(n_\mu)] \neq 0.$$

From above discussion, we have

$$\alpha + \beta + \delta = 0, \alpha + \beta + \lambda = 0, \alpha + \beta + \mu = 0, \lambda + \mu = \delta + \mu = \delta + \lambda,$$

and $\delta = \lambda = \mu = 0$. Contradiction. It follows that if $\alpha + \beta \neq 0$, then $(A_\alpha, A_\beta)Q\Gamma(A)_0 = 0$.

If $\alpha + \beta = 0$, then for all $h_1, h_2 \in T$ and $f_0 \in Q\Gamma(A)_0$,

$$(h_1, h_2)(x_\alpha, x_{-\alpha})f_0 = (x_\alpha, x_{-\alpha})(h_1, h_2)f_0 + [(h_1, h_2), (x_\alpha, x_{-\alpha})]f_0 = 0.$$

It implies that $(A_\alpha, A_{-\alpha})Q\Gamma(A)_0 \subseteq Q\Gamma(A)_0$. Thanks to Lemma 5.2 and 5.3, for all $f_0 \in \Gamma(A)_0$, $\delta \in \Omega$, $x_\delta \in A_\delta$,

$$(x_\alpha, x_{-\alpha})f_0(x_\delta) = (x_{-\alpha}, x_\delta)f_0(x_\alpha) = (x_\delta, x_\alpha)f_0(x_{-\alpha}) = 0.$$

Therefore, $((x_\alpha, x_{-\alpha})f_0)(A_1) = 0$. The result 4) follows.

From Lemma 5.2 and Lemma 5.3,

$(T, T)Q\Gamma(L)_0 = 0$, $((T, A_1)Q\Gamma(A)_0)(T) = ((T, T)Q\Gamma(A)_0)(A_1) = 0$. It follows result 5). The proof is completed. \square

Theorem 5.5 *Let A be a 3-Lie algebra with trivial centre, then $Q\Gamma(A) = \Gamma(A) \oplus Q\Gamma(A)_1$ with $Q\Gamma(A)_1 Q\Gamma(A)_1 = 0$.*

Proof First, we prove that $\Gamma(A) = Q\Gamma(A)_0$. It is clear that $\Gamma(A) \subseteq Q\Gamma(A)_0$. And from Lemma 5.2, for all $f_0 \in Q\Gamma(A)_0$ and $t_1, t_2 \in T$, $(t_1, t_2)f_0 = 0$. Then for all $x \in A$ and $t \in T$,

$$[t_1, t_2, f_0(x)] = f_0([t_1, t_2, x]), [t_1, t_2, f_0(t)] = f_0([t_1, t_2, t]) = 0.$$

Thanks to Theorem 5.4, for all $\alpha, \beta \in \Omega$, $x_\alpha \in A_\alpha$, $x_\beta \in A_\beta$, $x \in A$ and $f_0 \in Q\Gamma(A)_0$, if $\alpha + \beta \neq 0$, $((x_\alpha, x_\beta)f_0)(x) = 0$. Then

$$f_0([x_\alpha, x_\beta, x]) = [x_\alpha, x_\beta, f_0(x)] - ((x_\alpha, x_\beta)f_0)(x) = [x_\alpha, x_\beta, f_0(x)].$$

If $\alpha + \beta = 0$, then by Theorem 5.4, for all $t \in T$, $\delta \in \Omega$, $x_\delta \in A_\delta$, $((x_\alpha, x_{-\alpha})f_0)(x_\delta) = 0$. Then

$$f_0([x_\alpha, x_{-\alpha}, x_\delta]) = [x_\alpha, x_{-\alpha}, f_0(x_\delta)] - ((x_\alpha, x_{-\alpha})f_0)(x_\delta) = [x_\alpha, x_{-\alpha}, f_0(x_\delta)].$$

Since $((x_\alpha, x_{-\alpha})f_0)(t) \in T$,

$$[((x_\alpha, x_{-\alpha})f_0)(t), T, T] = 0, [((x_\alpha, x_{-\alpha})f_0)(t), x_\delta, A] = [t, ((x_\alpha, x_{-\alpha})f_0)(x_\delta), A] = 0.$$

It follows that $((x_\alpha, x_{-\alpha})f_0)(t) \in Z(A) = 0$, $((x_\alpha, x_{-\alpha})f_0)(T) = 0$. Therefore,

$$f_0([x_\alpha, x_{-\alpha}, t]) = [x_\alpha, x_{-\alpha}, f_0(t)] - ((x_\alpha, x_{-\alpha})f_0)(t) = [x_\alpha, x_{-\alpha}, f_0(t)].$$

Summarizing above discussion, we have $f_0 \in \Gamma(A)$, and $\Gamma(A) = Q\Gamma(A)_0$.

Thanks to Theorem 5.4, $Q\Gamma(A)_1 Q\Gamma(A)_1(A) \subseteq Q\Gamma(A)_1(T) = 0$. \square

Theorem 5.6 *Let A be a 3-Lie algebra with the decomposition $A = A_1 \oplus A_2$, $[A_1, A_2, A] = 0$. Then we have*

$$Q\Gamma(A) = Q\Gamma(A_1) + Q\Gamma(A_2) + \Gamma_1 + \Gamma_2,$$

where $\Gamma_i = \{f \in \text{Hom}(A_i, A_j) \mid f(A_i) \subseteq Z(A_j), 1 \leq i \neq j \leq 2\}$.

Proof It is clear that $Q\Gamma(A_1) + Q\Gamma(A_2) + \Gamma_1 + \Gamma_2 \subseteq Q\Gamma(A)$.

Let $p_i \in \text{Hom}(A, A_i)$, $i = 1, 2$ be projections, that is, for all $x = x_1 + x_2 \in A$, $x_i \in A_i$, $p_i(x) = x_i$, $i = 1, 2$.

For all $f \in Q\Gamma(A)$, define $f_i \in \text{Hom}(A_i, A_i)$, $i = 1, 2$, $f_3 \in \text{Hom}(A_1, A_2)$, $f_4 \in \text{Hom}(A_2, A_1)$ as follows

$$f_1(x) = p_1 f(x), \text{ for all } x \in A_1; \quad f_2(x) = p_2 f(x), \text{ for all } x \in A_2;$$

$$f_3(x) = p_2 f(x), \text{ for all } x \in A_1; \quad f_4(x) = p_1 f(x), \text{ for all } x \in A_2,$$

Without loss of generality, suppose $f_1(A_2) = f_3(A_2) = 0$, $f_2(A_1) = f_4(A_1) = 0$. Then for all $x = x_1 + x_2 \in A$, $x_1 \in A_1$ and $x_2 \in A_2$,

$$f(x) = f(x_1) + f(x_2) = p_1 f(x_1) + p_2 f(x_1) + p_1 f(x_2) + p_2 f(x_2) = (f_1 + f_2 + f_3 + f_4)(x).$$

Thanks to $[A, A_1, A_2] = 0$, $f_1 \in QC(A_1)$, $f_2 \in QC(A_2)$.

For all $x \in A_1$, $y, z \in A_2$, since

$$[f_3(x), y, z] = [f(x), y, z] = [x, f(y), z] = 0, \quad [f_3(A_1), A_2, A_2] = 0. \text{ It follows that } f_3 \in \Gamma_1.$$

Similarly, we have $f_4 \in \Gamma_2$. Therefore, $Q\Gamma(A) \subseteq Q\Gamma(A_1) + Q\Gamma(A_2) + \Gamma_1 + \Gamma_2$. The result follows. \square

Corollary 5.7 *Let A be a 3-Lie algebra with the decomposition $A = A_1 \oplus \cdots \oplus A_m$, and $[A_i, A_j, A] = 0$, $1 \leq i \neq j \leq m$. Then*

$$Q\Gamma(A) = Q\Gamma(A_1) + \cdots + Q\Gamma(A_m) + \sum_{1 \leq i \neq j \leq m} \Gamma_{ij},$$

where $\Gamma_{ij} = \{f \in \text{Hom}(A_i, A_j) \mid f(A_i) \subseteq Z(A_j), 1 \leq i \neq j \leq m\}$.

Proof The discussion is completely similar to Theorem 5.6. \square

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